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HÖLDER REGULARITY FOR NON-AUTONOMOUS ABSTRACT PARABOLIC EQUATIONS

BY

GIUSEPPE DA PRATO[†] AND EUGENIO SINESTRARI[†]

ABSTRACT

This paper contains some existence and uniqueness results for the strict and classical solutions $u:[0,T]\to E$ of the non-autonomous evolution equation $u'(t)=\Lambda(t)u(t)+f(t)$ in a Banach space E under the classical Tanabe–Sobolevski assumptions. These results do not require use of the fundamental solution and give new information about the hölder-regularity of the solutions.

0. Introduction

The present paper is concerned with the strict solutions $u:[0,T]\to E$ of the following time-dependent abstract evolution equation

$$\text{(P)}\qquad \begin{cases} u'(t)=\Lambda(t)u(t)+f(t), & 0\leqq t\leqq T \\ u(0)=x \end{cases}$$

where, for each $t\in[0,T]$, $\Lambda(t)$ is the infinitesimal generator of an analytic semigroup in a Banach space E. The domain of $\Lambda(t)$, $D_{\Lambda(t)}$, is supposed to be independent of t and $t\to\Lambda(t)$ to be hölder continuous from $[0,T]$ to $\mathscr{L}(D_{\Lambda(0)},E)$. With equivalent assumptions problem (P) was studied by Tanabe [18], Sobolevski [16] and Poulsen [12] (see also Friedman [5], Ladas–Lakshmikantham [8] and Tanabe [19]): they constructed the fundamental solution for (P) and proved that if f is hölder continuous then there exists a strict solution of (P). In this work we do not need to introduce the fundamental solution because we use some sharp regularity results for equation (P) in the autonomous case (see for example Sinestrari [14]) to prove the existence of a strict solution u whose derivative is hölder continuous in $[0,T]$: this method

[†] Work done as a member of G.N.A.F.A. of C.N.R.

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seems to be more simple and gives new results (see comments before Theorems 4.2 and 4.3); moreover it enables us to extend these results to the non-linear case, as will be done in a subsequent paper. Problem (P) has been considered in L^p spaces by Da Prato–Grisvard [2] by a method which cannot be directly extended to our situation.

In section 1 we state the exact assumptions on the operators $\Lambda(t)$: some consequences of these hypotheses are derived in the appendix (at the end of the paper) along with the proof of their equivalence with Tanabe's and Sobolevski's assumptions.

In section 2 some intermediate spaces between D_Λ and E are introduced to state the above-mentioned sharp regularity results (in the hölder spaces) for the solutions of (P) in the autonomous case.

In section 3 we define the strict solution and the classical solution of (P) and we demonstrate a uniqueness theorem. The proof of the existence of the strict and of the classical solution is given in section 4. In the appendix are given additional properties of $\Lambda(t)$ which will be also needed in the future developments of the results contained in this paper.

1. Preliminaries

We give now some definitions and notation which will be used later. Let E be a Banach space with norm $\|\cdot\|$: if $\Lambda : D_\Lambda \subseteq E \to E$ is a closed linear operator, D_Λ will be always endowed with the graph norm $\|x\|_{D_\Lambda} = \|x\| + \|\Lambda x\|$. Let F be another Banach space with norm $\|\cdot\|_F$, continuously embedded in E (i.e. $F \hookrightarrow E$). We shall make the following assumptions:

(I) *for each $t \in [0, T]$, $\Lambda(t): D_{\Lambda(t)} \subseteq E \to E$ is the infinitesimal generator of an analytic semigroup in E.*

(II) *for each $t \in [0, T]$, $D_{\Lambda(t)} \simeq F$; i.e. $D_{\Lambda(t)} = F$ and the norm $\|\cdot\|_{D_{\Lambda(t)}}$ is equivalent to $\|\cdot\|_F$.*

From this it follows that $\Lambda(t) \in \mathscr{L}(F, E)$. We shall also assume that:

(III) *$\Lambda : t \to \Lambda(t)$ belongs to $C^\alpha(0, T; \mathscr{L}(F, E))$, i.e. is hölder continuous with exponent $\alpha \in]0, 1[$.*

By substituting (if necessary) the unknown function of (P) with $e^{-\delta t}u(t)$ with suitable $\delta \in \mathbf{R}$ we obtain a problem which is equivalent to (P) (with respect to the properties which will interest us): hence, as will be seen in Proposition A.5 of the appendix, we can suppose that:

(IV) *there exist* $\omega < 0$, $\theta \in]\pi/2, \pi]$ *and* $M > 0$ *such that if* $\lambda \in \omega + S_\theta = \{\omega + z, z \in \mathbf{C} - \{0\}, |\arg z| < \theta\}$ *and* $t \in [0, T]$ *then we have* $\lambda \in \rho(\Lambda(t))$ *and*

$$\|R(\lambda, \Lambda(t))\|_{\mathcal{L}(E)} \leqq \frac{M}{|\lambda - \omega|} \tag{1.1}$$

where $\rho(\Lambda(t))$ is the resolvent set of $\Lambda(t)$ and $R(\lambda, \Lambda(t)) = (\lambda - \Lambda(t))^{-1}$.

If $s \to e^{\Lambda(t)s}$ is the analytic semigroup generated by $\Lambda(t)$ from (1.1) it follows (see Proposition A.6) that there exist M_k $(k = 0, 1, 2, \cdots)$ such that for each $t \in [0; T]$

$$\|s^k \Lambda^k(t) e^{\Lambda(t)s}\|_{\mathcal{L}(E)} \leqq M_k \qquad \text{for } s \geqq 0. \tag{1.2}$$

Let us introduce now some notation about the vector valued function spaces which we shall need.

Let E be a Banach space with norm $\|\cdot\|$ and α, a, b real numbers such that $0 < \alpha < 1$ and $a < b$.

$C(a, b; E)$ is the Banach space of continuous functions

$$u : [a, b] \to E \quad \text{with norm } \|u\|_{C(a,b;E)} = \sup\{\|u(t)\|, a \leqq t \leqq b\},$$

$C^1(a, b; E) = \{u \in C(a, b; E), u' \text{ exists in } C(a, b; E)\}$.

$C^\alpha(a, b; E)$ is the Banach space of hölder continuous functions $u : [a, b] \to E$ with exponent α and norm given by

$$\|u\|_{C(a,b;E)} + \sup\left\{\frac{\|u(t) - u(s)\|}{|t - s|^\alpha}, t, s \in [a, b], t \neq s\right\}.$$

$h^\alpha(a, b; E)$ is the subspace of $u \in C^\alpha(a, b; E)$ satisfying the following condition: for each $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that if $|t - s| < \delta_\varepsilon$ then $\|u(t) - u(s)\| \leqq \varepsilon |t - s|^\alpha$. It can be seen that $h^\alpha(a, b; E)$ is the completion of $C^1(a, b; E)$ in $C^\alpha(a, b; E)$. For more details on h^α spaces see Kufner–John–Fucik [7].

$C^{1,\alpha}(a, b; E) = \{u \in C^1(a, b; E);\ u' \in C^\alpha(a, b; E)\}$. We give a similar definition for $h^{1,\alpha}(a, b; E)$.

$C_0(a, b; E) = \{u \in C(a, b; E), u(a) = 0\}$.

$C_0^1(a, b; E) = \{u \in C^1(a, b; E), u(a) = 0\}$.

$C(a^+, b; E) = \{u \in C(\varepsilon, b; E)$ for each $\varepsilon \in]0, b - a[\}$; a similar definition is given for $C^1(a^+, b; E)$, $C^\alpha(a^+, b; E)$ and $C^{1,\alpha}(a^+, b; E)$.

2. Maximal regularity theorems in hölder spaces

Let us consider now (P) in the autonomous case:

$$\begin{cases} u'(t) = \Lambda u(t) + f(t), \\ u(0) = x. \end{cases} \tag{2.1}$$

Here $\Lambda : D_\Lambda \subseteq E \to E$ is the infinitesimal generator of a bounded analytic semigroup $\{e^{\Lambda t}\}$ and $f \in C^\theta(0, T; E)$; we know that (see e.g. Kato [6], Pazy [11])

$$u(t) = e^{\Lambda t}x + \int_0^t e^{\Lambda(t-s)} f(s)ds, \qquad 0 \leqq t \leqq T \tag{2.2}$$

gives the unique $u \in C(0, T, E)$ such that (2.1) is verified for $0 < t \leqq T$. By introducing some intermediate spaces (between D_Λ and E) it is possible to give conditions on x and $f(0)$ which ensure that u verifies (2.1) in $[0, T]$ and moreover $u' \in C^\theta(0, T; E)$ (by taking $f(t) = \text{const}$, it can be seen that this property does not hold without conditions on x and $f(0)$).

Definition 2.1. For each $0 < \theta < 1$ let $D_\Lambda(\theta, \infty)$ be defined as

$$D_\Lambda(\theta, \infty) = \left\{ x \in E, \sup_{t>0} \| t^{-\theta}(e^{\Lambda t}x - x) \| < \infty \right\}$$

with norm

$$\| x \|_\theta = \| x \| + \sup_{t>0} \| t^{-\theta}(e^{\Lambda t}x - x) \|.$$

The function $t \to e^{\Lambda t}x$ is in $C^\theta(0, T; E)$ for some $T > 0$ if and only if $x \in D_\Lambda(\theta, \infty)$.

The closure of D_Λ in $D_\Lambda(\theta, \infty)$ can be characterized as the subspace of E defined in the following way (see Butzer–Berens [1]):

Definition 2.2.

$$D_\Lambda(\theta) = \left\{ x \in E, \lim_{t \to 0} \| t^{-\theta}(e^{\Lambda t}x - x) \| = 0 \right\}.$$

The function $t \to e^{\Lambda t}x$ is in $h^\theta(0, T; E)$ for some $T > 0$ if and only if $x \in D_\Lambda(\theta)$.

The space $D_\Lambda(\theta, \infty)$ can be considered as a real interpolation space between D_Λ and E (the mean's space $S(\infty, \xi_0, D_\Lambda; \infty, \xi_1, E)$ with $1 - \theta = \xi_0(\xi_0 - \xi_1)^{-1}$ according to the definition of Lions–Peetre [9]) and $D_\Lambda(\theta)$ as a continuous interpolation space according to the definition of Da Prato–Grisvard [3] (see also Sinestrari–Vernole [13]). The following properties are proved in Butzer–Berens [1]:

$$
(2.3) \qquad \begin{cases} D_\Lambda(\theta,\infty) = \left\{ x \in E\,;\ \sup_{t>0} \| t^{1-\theta}\Lambda e^{\Lambda t}x \| < \infty \right\}, \\ D_\Lambda(\theta) = \left\{ x \in E\,;\ \lim_{t\to 0} \| t^{1-\theta}\Lambda e^{\Lambda t}x \| = 0 \right\}. \end{cases}
$$

We can write now the above-mentioned results about the maximal regularity of the solutions of (2.1).

THEOREM 2.3. *Let $\Lambda : D_\Lambda \subseteq E \to E$ be the infinitesimal generator of an analytic semigroup $\{e^{\Lambda t}\}$ satisfying*

$$
\| e^{\Lambda t} \|_{\mathscr{L}(E)} \leqq M_0 \quad \text{and} \quad \| t\Lambda e^{\Lambda t} \|_{\mathscr{L}(E)} \leqq M_1, \qquad \text{for } t \geqq 0.
$$

Then, setting

$$
(2.4) \qquad (e^\Lambda * f)(t) = \int_0^t e^{\Lambda(t-s)} f(s)ds, \qquad 0 \leqq t \leqq T
$$

we have the following results:

(i) *If $f \in C_0^\theta(0,T;E)$ and $x = 0$, then $e^\Lambda * f$ is a solution of (2.1) in $[0,T]$ satisfying $e^\Lambda * f \in C^\theta(0,T;D_\Lambda) \cap C^{1,\theta}(0,T;E)$. Moreover there is $K_1 > 0$ (depending on M_0, M_1 and T) such that for every $f \in C_0^\theta(0,T;E)$*

$$
(2.5) \qquad \| e^\Lambda * f \|_{C^\theta(0,T;D_\Lambda)} \leqq K_1 \| f \|_{C^\theta(0,T;E)}.
$$

(ii) *If $f \in C^\theta(0,T;E)$, $x \in D_\Lambda$ and $\Lambda x + f(0) \in D_\Lambda(\theta,\infty)$ then the function u given by (2.2) is a solution of (2.1) in $[0,T]$ satisfying $u \in C^\theta(0,T;D_\Lambda) \cap C^{1,\theta}(0,T;E)$; if moreover $f \in h^\theta(0,T;E)$ and $\Lambda x + f(0) \in D_\Lambda(\theta)$ then $u \in h^\theta(0,T;D_\Lambda) \cap h^{1,\theta}(0,T;E)$.*

PROOF. In Da Prato–Grisvard [2] it is shown that $e^\Lambda * f \in C^{1,\theta}(0,T;E)$ and in Sinestrari [14] the results of (ii) are proved.

REMARK 2.4. Theorem 2.3 holds with obvious modifications if $[0,T]$ is replaced by $[t_1,t_2] \subseteq \mathbf{R}_+$.

REMARK 2.5. We call these regularity results "of maximal type" as they prove that if f belongs to a suitable subspace X ($= C^\theta(0,T;E)$ or $h^\theta(0,T;E)$) of $C(0,T;E)$ then the solution of (2.1) exists and moreover u', $\Lambda u \in X$ (provided x and $f(0)$ satisfy suitable conditions). As will be seen in the next sections, this kind of regularity is necessary to prove our theorems in the non-autonomous case.

3. Statement of the problem and uniqueness results

We shall study problem (P) under assumptions (I)–(IV) of section 1.

DEFINITION 3.1. Let $f \in C(0, T; E)$. A function $u \in C(0, T; F) \cap C^1(0, T; E)$ is a strict solution of (P) if $u(0) = x$ and $u'(t) = \Lambda(t)u(t) + f(t)$ for $0 \leqq t \leqq T$.

DEFINITION 3.2. Let $f \in C(0, T; E)$. A function $u \in C(0, T; E) \cap C(0^+, T; F) \cap C^1(0^+, T; E)$ is a classical solution of (P) if $u(0) = x$ and $u'(t) = \Lambda(t)u(t) + f(t)$ for $0 < t \leqq T$.

REMARK 3.3. Definition 3.1 is equivalent to the condition that (P) can be considered as an equation in the function space $C(0, T; E)$: in fact if (P) holds in $[0, T]$ and $f, u' \in C(0, T; E)$ we have necessarily $u \in C(0, T; F)$. This can be seen by writing $u(t) = \Lambda^{-1}(t)[u'(t) - f(t)]$ and taking into account the fact that $\Lambda^{-1} \in C(0, T; \mathcal{L}(E, F))$ (see Proposition A.7). Definition 3.2 is usual in the parabolic equations when no restriction is imposed on the initial datum x (see Tanabe [19] pp. 69 and 127).

It is possible to prove the uniqueness of the classical (and hence of the strict) solution of (P). More precisely the following result holds:

THEOREM 3.4. *Let* $0 \leqq t_0 < T$. *If the function* $u \in C(t_0, T; E) \cap C(t_0^+, T; F) \cap C^1(t_0^+, T; E)$ *verifies*

$$\begin{cases} u'(t) = \Lambda(t)u(t), & t_0 < t \leqq T \\ u(t_0) = 0 \end{cases} \tag{3.1}$$

then $u \equiv 0$ *on* $[t_0, T]$.

This theorem will be demonstrated in a subsequent paper; as in section 4 we will prove the existence of a classical (and of a strict) solution verifying the following condition (see (4.6)): for each $t_1 \in [0, T]$ we have

$$\sup_{t_1 < t \leqq T} \|(t - t_1)^{1-\beta} \Lambda(t)u(t)\| < \infty \tag{3.2}$$

with $0 < 1 - \beta < \alpha$, then we will prove Theorem 3.4 here under the additional assumption (3.2) on u.

PROOF. Let us suppose by contradiction that $t_1 = \sup\{t; u \equiv 0 \text{ on } [t_0, t]\}$ is less than T. From the definition of t_1 we have that for each $\delta \in]0, T - t_1]$,

$$m = \sup_{t_1 < t \leqq t_1 + \delta} \|w(t)\| > 0$$

where

$$w(t) = (t - t_1)^{1-\beta}\Lambda(t)u(t), \qquad t_1 < t \leqq t_1 + \delta.$$

Let us take $t \in]t_1, t_1 + \delta]$ and then $\varepsilon \in]0, t - t_1[$. The function $g(s) = e^{\Lambda(t)(t-s)}u(s)$ is continuously differentiable in $[t_1 + \varepsilon, t]$ hence from $g(t) - g(t_1 + \varepsilon) = \int_{t_1+\varepsilon}^{t} g'(s)ds$ and from (3.1) we get

$$u(t) - e^{\Lambda(t)(t-t_1-\varepsilon)}u(t_1 + \varepsilon) = \int_{t_1+\varepsilon}^{t} e^{\Lambda(t)(t-s)}[\Lambda(s) - \Lambda(t)]u(s)ds.$$

From this it follows that

$$w(t) - (t - t_1)^{1-\beta}\Lambda(t)e^{\Lambda(t)(t-t_1-\varepsilon)}u(t_1 + \varepsilon)$$
$$= (t - t_1)^{1-\beta}\int_{t_1+\varepsilon}^{t} \Lambda(t)e^{\Lambda(t)(t-s)}[1 - \Lambda(t)\Lambda^{-1}(s)](s - t_1)^{\beta-1}w(s)ds$$

and hence for $\varepsilon \to 0$

$$w(t) = (t - t_1)^{1-\beta}\int_{t_1}^{t} \Lambda(t)e^{\Lambda(t)(t-s)}[1 - \Lambda(t)\Lambda^{-1}(s)](s - t_1)^{\beta-1}w(s)ds.$$

From (A.9) and $(\mathrm{A}.11)_1$ of the appendix we deduce for each $t \in]t_1, t_1 + \delta]$

$$\|w(t)\| \leqq mK_1M_1(t - t_1)^{1-\beta}\int_{t_1}^{t} (t - s)^{\alpha-1}(s - t_1)^{\beta-1}ds$$
$$= mK_1M_1(t - t_1)^{\alpha}\int_0^1 (1 - \sigma)^{\alpha-1}\sigma^{\beta-1}d\sigma$$

so that

$$m \leqq mK_1M_1\delta^{\alpha}\int_0^1 (1 - \sigma)^{\alpha-1}\sigma^{\beta-1}d\sigma.$$

As $m > 0$, for δ sufficiently small we get a contradiction.

4. Existence of the strict and the classical solution

In this section we will prove our main results, i.e. the existence of strict and classical solutions of (P) under suitable conditions on f and x.

To prove these results we need the following lemma:

LEMMA 4.1. *Let* $\Lambda : t \to \Lambda(t)$ *verify* (I)–(IV) *of section* 1. *Setting*

$$F(t) = (\Lambda(t) - \Lambda(t_0))e^{\Lambda(t_0)(t-t_0)}x \qquad (0 \leqq t_0 \leqq t \leqq T) \tag{4.1}$$

where

(4.2) $$x \in D_{\Lambda(t_0)}(\beta,\infty) \quad \textit{and} \quad 0 \leqq 1-\beta < \alpha < 1$$

(we set $D_{\Lambda(t_0)}(1,\infty) = D_{\Lambda(t_0)}$*) we have*

(4.3) $$F \in C_0^{\alpha+\beta-1}(t_0, T; E).$$

If moreover $\Lambda \in h^\alpha(0,T;\mathscr{L}(F,E))$ *we have*

(4.4) $$F \in h_0^{\alpha+\beta-1}(t_0, T; E).$$

PROOF. From (4.2) and (2.3) there exists $C_0 > 0$ such that

$$\| t^{1-\beta}\Lambda(t_0)e^{\Lambda(t_0)t}x\| \leqq C_0, \qquad t \geqq 0.$$

By writing $t^{2-\beta}\Lambda^2(t_0)e^{\Lambda(t_0)t}x = t^{2-\beta}\Lambda(t_0)e^{\Lambda(t_0)t/2}\Lambda(t_0)e^{\Lambda(t_0)t/2}x$ we deduce the existence of C_0' such that for each $t \geqq 0$:

$$\| t^{2-\beta}\Lambda^2(t_0)e^{\Lambda(t_0)t}x\| \leqq C_0', \qquad t \geqq 0.$$

Let M_0, M_0', $k > 0$ satisfy:

$$\| e^{\Lambda(t_0)t}\|_{\mathscr{L}(E)} \leqq M_0, \quad \| t\Lambda(t_0)e^{\Lambda(t_0)t}\|_{\mathscr{L}(E)} \leqq M_0', \qquad t \geqq 0,$$

$$\| \Lambda(t) - \Lambda(s)\|_{\mathscr{L}(F,E)} \leqq k\,|t-s|^\alpha, \qquad t,s \in [0,T].$$

For $t_0 \leqq s < t \leqq T$ we can write:

$$F(t) - F(s) = (\Lambda(t) - \Lambda(s))e^{\Lambda(t_0)(t-t_0)}x + (\Lambda(s) - \Lambda(t_0))(e^{\Lambda(t_0)(t-t_0)}x - e^{\Lambda(t_0)(s-t_0)}x)$$
$$= I_1 + I_2.$$

If $\|\cdot\|_F \leqq \alpha_0\|\cdot\|_{D_{\Lambda(t_0)}}$, we get

$$\| I_1\| \leqq k(t-s)^\alpha\alpha_0(M_0 + C_0(t-t_0)^{\beta-1})\| x\| \leqq \text{const}(t-s)^{\alpha+\beta-1}$$

and (when $t_0 < s$)

$$\| I_2\| \leqq k(s-t_0)^\alpha\alpha_0\| e^{\Lambda(t_0)(t-t_0)}x - e^{\Lambda(t_0)(s-t_0)}x\|_{D_{\Lambda(t_0)}}.$$

Now

$$\| e^{\Lambda(t_0)(t-t_0)}x - e^{\Lambda(t_0)(s-t_0)}x\|_{D_{\Lambda(t_0)}} = \left\| \int_s^t \Lambda(t_0)e^{\Lambda(t_0)(\sigma-t_0)}x\,d\sigma \right\|_{D_{\Lambda(t_0)}}$$
$$= \left\| \int_s^t \Lambda(t_0)e^{\Lambda(t_0)(\sigma-t_0)}x\,d\sigma \right\| + \left\| \int_s^t \Lambda^2(t_0)e^{\Lambda(t_0)(\sigma-t_0)}x\,d\sigma \right\|$$
$$\leqq C_0\int_s^t (\sigma-t_0)^{\beta-1}d\sigma + C_0'\int_s^t (\sigma-t_0)^{\beta-2}d\sigma$$

and hence

$$\|I_2\| \leqq k(s-t_0)^\alpha \alpha_0(C_0T+C_0')\int_s^t (\sigma-t_0)^{\beta-2}d\sigma \leqq \text{const}\int_s^t (\sigma-t_0)^{\alpha+\beta-2}d\sigma$$

$$\leqq \text{const}(t-s)^{\alpha+\beta-1}.$$

This proves (4.3). The proof of (4.4) is similar. □

Under assumptions equivalent to ours (see Proposition A.8) and when $x \in F$ and $f \in C^\theta(0,T;E)$, Tanabe [18] and Sobolevski [16] proved the existence of a strict solution u. Later Poulsen [12] demonstrated that $u' \in C^\beta(0^+,T:E)$ with $\beta < \min(\alpha,\theta)$ (which was previously shown by Tanabe [17] in the case $\alpha = 1$) and $u' \in C^\theta(0,T;E)$ if $x = f(0) = 0$.

With the aid of the preceding lemma we can add new results about the strict solution:

THEOREM 4.2. *Let* Λ *verify* (I)–(IV) *of section* 1 *and let in addition* $\Lambda \in h^\alpha(0,T;\mathscr{L}(F,E))$. *If we take* $f \in C^\alpha(0,T;E)$ *and* $x \in F$ *then there exists a unique strict solution to* (P). *More precisely*:

(i) *If* $f \in C^\alpha(0,T;E)$, $\Lambda \in h^\alpha(0,T;\mathscr{L}(F,E))$ *and* $x \in F$, *then there exists in* $[0,T]$ *a unique strict solution* u *of* (P) *such that* $u \in C^\alpha(0^+,T;F)\cap C^{1,\alpha}(0^+,T;E)$; *if moreover* $\Lambda(0)x+f(0)\in D_{\Lambda(0)}(\alpha,\infty)$ *then* $u \in C^\alpha(0,T;F)\cap C^{1,\alpha}(0,T;E)$.

(ii) *If in addition* $f \in h^\alpha(0^+,T;E)$, *then* $u \in h^\alpha(0^+,T;F)\cap h^{1,\alpha}(0^+,T;E)$; *if moreover* $\Lambda(0)x+f(0)\in D_{\Lambda(0)}(\alpha)$ *then* $u \in h^\alpha(0,T;F)\cap h^{1,\alpha}(0,T;E)$.

PROOF. For fixed $t_0 \in [0,T[$ and $x_0 \in F$, the problem

$$\text{(Z)}\qquad \begin{cases} z'(t) = \Lambda(t_0)z(t)+f(t_0), & t_0 \leqq t \leqq T \\ z(t_0) = x_0 \end{cases}$$

has a strict solution given by

$$z(t) = e^{\Lambda(t_0)(t-t_0)}(x_0+\Lambda^{-1}(t_0)f(t_0))-\Lambda^{-1}(t_0)f(t_0). \tag{4.5}$$

Hence

$$z \in C^\alpha(t_0^+,T;F)\cap C^{1,\alpha}(t_0^+,T;E).$$

Now if a function $v \in C^\alpha(t_0,t_1;F)\cap C^{1,\alpha}(t_0,t_1;E)$ $(0 \leqq t_0 < t_1 \leqq T)$ is a strict solution of

$$\text{(V)}\qquad \begin{cases} v'(t) = \Lambda(t_0)v(t)+[\Lambda(t)-\Lambda(t_0)]v(t)+[\Lambda(t)-\Lambda(t_0)]z(t)+f(t)-f(t_0), \\ \hfill t_0 \leqq t \leqq t_1 \\ v(t_0) = 0 \end{cases}$$

then, setting

$$u(t) = v(t) + z(t),$$

we have that u belongs to the space

$$C^\alpha(t_0^+, t_1; F) \cap C^{1,\alpha}(t_0^+, t_1; E)$$

and is a strict solution of

$$(\mathrm{P}_0) \qquad \begin{cases} u'(t) = \Lambda(t)u(t) + f(t), & t_0 \leqq t \leqq t_1, \\ u(t_0) = x_0. \end{cases}$$

Let us set for $t_0 \leqq t \leqq T$

$$\varphi(t) = [\Lambda(t) - \Lambda(t_0)]z(t) + f(t) - f(t_0).$$

As $x_0 \in F$, $f \in C^\alpha(0, T; E)$ and $\Lambda \in C^\alpha(0, T; \mathscr{L}(F, E))$, taking $\beta = 1$ in Lemma 4.1 we have $\varphi \in C_0^\alpha(t_0, T; E)$.

Setting $X = C^\alpha(t_0, t_1; F)$, let us define for each $v \in X$

$$S(v)(t) = \int_{t_0}^{t} e^{\Lambda(t_0)(t-s)}[\Lambda(s) - \Lambda(t_0)]v(s)ds \qquad (t_0 \leqq t \leqq t_1)$$

and set

$$v_0(t) = \int_{t_0}^{t} e^{\Lambda(t_0)(t-s)}\varphi(s)ds \qquad (t_0 \leqq t \leqq t_1).$$

As $[\Lambda(\cdot) - \Lambda(t_0)]v(\cdot) \in C_0^\alpha(t_0, t_1; E)$, from Theorem 2.3 we deduce that v_0, $S(v) \in X$. From the same theorem we have that if $v \in X$ verifies

$$(\mathrm{S}) \qquad v = S(v) + v_0$$

then v belongs to $C^\alpha(t_0, t_1; E) \cap C^{1,\alpha}(t_0, t_1; E)$ and is a strict solution to (V). Now for each $v \in X$

$$\begin{aligned} \|S(v)\|_X &= \|e^{\Lambda(t_0)} * [\Lambda(\cdot) - \Lambda(t_0)]v(\cdot)\|_X \\ &= K_1 \|\Lambda(\cdot) - \Lambda(t_0)\|_{C^\alpha(t_0, t_1; \mathscr{L}(F,E))} \cdot \|v\|_X \end{aligned}$$

with K_1 independent of t_0, t_1 and v (see (i) of Theorem 2.3). As $\Lambda \in h^\alpha(0, T; \mathscr{L}(F, E))$, there exists $\delta > 0$ such that if $|t_1 - t_0| < \delta$ then $\|S\|_{\mathscr{L}(X)} < 1$. Hence (V) can be solved in $[t_0, t_1]$ with $t_1 = \inf(t_0 + \delta, T)$ and δ independent of t_0 and x_0; so that the problem (P_0) has a solution $u \in C^\alpha(t_0^+, t_1; F) \cap C^{1,\alpha}(t_0^+, t_1; E)$.

If we take $t_0 = 0$ and $x_0 = x$ we obtain a strict solution u of (P) in $[0, \inf(\delta, T)]$. If in addition $\Lambda(0)x + f(0) \in D_{\Lambda(0)}(\alpha, \infty)$, then $u \in C^\alpha(0, \inf(\delta, T); F) \cap$

$C^{1,\alpha}(0, \inf(\delta, T); E)$. Now if $\delta < T$, we can set $t_0 = \delta$, $x_0 = u(\delta) \in F$ and extend u to a solution of (P) in $[0, 2\delta] \cap [0, T]$: proceeding in this way, we get a strict solution u of (P) in $[0, T]$. As t_0 can be chosen arbitrarily in $[0, T[$ and δ is independent of t_0, then $u \in C^\alpha(0^+, T; F) \cap C^{1,\alpha}(0^+, T; E)$ and (i) is proved.

To obtain (ii) it is sufficient to replace in the preceding proof C^α, $C^{1,\alpha}$ and $D_{\Lambda(0)}(\alpha, \infty)$ by h^α, $h^{1,\alpha}$ and $D_{\Lambda(0)}(\alpha)$ respectively and to use (ii) of Theorem 2.3. □

When $f \in C^\theta(0, T; E)$ and $x \in E$, Tanabe [18] and Sobolevski [16] proved (with the aid of the fundamental solution) the existence of a classical solution. We will prove this result with a different method in a future paper; here we consider only the case when x belongs to a suitable intermediate space between F and E. In this case we obtain new information about the regularity of the classical solution. More precisely we prove the following result about the classical solution of (P).

THEOREM 4.3. *Let Λ verify* (I)–(IV) *of section* 1. *Let $0 < \theta < \alpha < 1$ and set $\beta = \theta - \alpha + 1$. If $x \in D_{\Lambda(0)}(\beta, \infty)$ and $f \in C^\theta(0, T; E)$, then there exists in $[0, T]$ a unique classical solution u of* (P) *such that $u \in C^\beta(0, T; E) \cap C^\theta(0^+, T; F) \cap C^{1,\theta}(0^+, T; E)$. If in addition $x \in D_{\Lambda(0)}(\beta)$, $\Lambda \in h^\alpha(0, T; \mathscr{L}(F, E))$ and $f \in h^\theta(0, T; E)$ then $u \in h^\beta(0, T; E) \cap h^\theta(0^+, T; F) \cap h^{1,\theta}(0^+, T; E)$.*

PROOF. We proceed as in the proof of the previous theorem. The classical solution of the problem

$$(\mathrm{Z}_0) \qquad \begin{cases} z'(t) = \Lambda(0)z(t) + f(0), & 0 \leqq t \leqq T \\ z(0) = x \end{cases}$$

is given by $z(t) = e^{\Lambda(0)t}(x + \Lambda^{-1}(0)f(0)) - \Lambda^{-1}(0)f(0)$. As $x \in D_{\Lambda(0)}(\beta, \infty)$ we have $z \in C^\beta(0, T; E)$.

If $v \in C^\theta(0, \delta; F) \cap C^{1,\theta}(0, \delta; E)$ $(0 < \delta \leqq T)$ is a strict solution of

$$(\mathrm{V}_0) \qquad \begin{cases} v'(t) = \Lambda(0)v(t) + [\Lambda(t) - \Lambda(0)]v(t) + [\Lambda(t) - \Lambda(0)]z(t) + f(t) - f(0) \\ v(0) = 0 \end{cases}$$

then setting for $t \in [0, \delta]$, $u(t) = v(t) + z(t)$ we have that u belongs to $C^\beta(0, \delta; E) \cap C^\theta(0^+, \delta; F) \cap C^{1,\theta}(0^+, \delta; E)$ and is a classical solution of

$$(\mathrm{P}_0) \qquad \begin{cases} u'(t) = \Lambda(t)u(t) + f(t), & 0 \leqq t \leqq \delta, \\ u(0) = x. \end{cases}$$

If we set for $t \in [0, T]$:

$$\varphi(t)=[\Lambda(t)-\Lambda(0)]z(t)+f(t)-f(0)$$

we deduce from Lemma 4.1 that $\varphi\in C_0^\theta(0,T;E)$.

Setting $X=C^\theta(0,\delta;F)$ let us define for each $v\in X$

$$S(v)(t)=\int_0^t e^{\Lambda(0)(t-s)}[\Lambda(s)-\Lambda(0)]v(s)ds,\qquad 0\leqq t\leqq\delta$$

and set

$$v_0(t)=\int_0^t e^{\Lambda(0)(t-s)}\varphi(s)ds,\qquad 0\leqq t\leqq\delta.$$

As $[\Lambda(\cdot)-\Lambda(0)]v(\cdot)\in C_0^\theta(0,\delta;E)$ from Theorem 2.3 we deduce that v_0, $S(v)\in X$. From the same theorem we have that if $v\in X$ verifies

$$(S_0)\qquad v=S(v)+v_0$$

then v belongs to $C^\theta(0,\delta;F)\cap C^{1,\theta}(0,\delta;E)$ and is a strict solution of (V_0).

Now for each $v\in X$:

$$\begin{aligned}\|S(v)\|_X&\leqq\|e^{\Lambda(0)}*[\Lambda(\cdot)-\Lambda(0)]v(\cdot)\|_X\\&\leqq K_2\|\Lambda(\cdot)-\Lambda(0)\|_{C^\theta(0,\delta;\mathscr{L}(F,E))}\|v\|_X\end{aligned}$$

with K_2 independent of t_0, t_1 and v (see (i) of Theorem 2.3). As $\Lambda\in C^\alpha(0,T;\mathscr{L}(F,E))\subset h^\theta(0,T;\mathscr{L}(F,E))$, for sufficiently small δ $(<T)$, we have $\|S\|_{\mathscr{L}(X)}<1$ and problem (P_0) has a classical solution u on $[0,\delta]$ such that $u\in C^\beta(0,\delta;E)$. To complete the proof of the first part of the theorem, as $u(t)\in F$ for $0<t\leqq\delta$, it is sufficient to apply Theorem 4.2 (with α replaced by θ) to problem (P_0) with $0<t_0\leqq\delta$ and $u(t_0)=x_0$. The last part of the theorem can be obtained from the preceding proof with the same modifications indicated at the end of the proof of Theorem 4.2.

Remark 4.4. The classical solution of Theorem 4.3 verifies the following property: for each $t_1\in[0,T[$ we have

$$(4.6)\qquad \sup_{t_1<t\leqq T}\|(t-t_1)^{1-\beta}\Lambda(t)u(t)\|<\infty.$$

To prove this let us use the notation of the preceding proof: as $u\in C^\theta(0^+,T;F)$ and (III) holds, it is sufficient to take $t_1=0$ and to show that $\|t^{1-\beta}\Lambda(t)u(t)\|$ is bounded in $]0,\delta]$; when $t\in]0,\delta]$ we can write $u(t)=v(t)+z(t)$ with $v\in C^\theta(0,\delta;F)$ and $z(t)=e^{\Lambda(0)t}(x+x_1)+x_1$ where $x\in D_{\Lambda(0)}(\beta,\infty)$ and $x_1\in F$. Now $t^{1-\beta}\Lambda(t)e^{\Lambda(0)t}x=\Lambda(t)\Lambda^{-1}(0)t^{1-\beta}\Lambda(0)e^{\Lambda(0)t}x$ and $\|\Lambda(t)\Lambda^{-1}(0)\|_{\mathscr{L}(E)}$ is bounded (see

(A.4) in the appendix), so that we have $\sup_{0<t\leqq\delta} \| t^{1-\beta}\Lambda(t)e^{\Lambda(0)t}x\| < \infty$ by virtue of (2.3). As $v(t)+e^{\Lambda(0)t}x_1+x_1$ is in $C(0,\delta;F)$ and (III) holds, the conclusion follows.

REMARK 4.5. By comparing (i) and (ii) of Theorem 4.2 we deduce that the subspace in which there is the maximal regularity for (P) is h^α and not C^α: in fact when $f, \Lambda \in h^\alpha$ we can deduce $u' \in h^\alpha$, i.e. all the terms of equation (P) belong to the same subspace of $C(0,T;E)$.

Appendix

In this section we want to show how one can deduce from conditions (I)–(III) on $t \to \Lambda(t)$ several propositions which enable us to prove that the resolvent of $\Lambda(t)$ contains a sector independent of t and that our conditions are equivalent to those of Tanabe [18] and Sobolevski [16]. Assumptions (I)–(III) can be rewritten in the following form:

Let $F \hookrightarrow E$ be Banach spaces with norm $\|\cdot\|_F$ and $\|\cdot\|$ respectively:

(I) *for each $t\in[0,T]$, $\Lambda(t): D_{\Lambda(t)} \subseteq E \to E$ is the infinitesimal generator of an analytic semigroup, i.e. $D_{\Lambda(t)}$ is dense in E and there are $\omega_t \in \mathbf{R}$, $M_t>0$, $\theta_t \in]\pi/2,\pi]$ such that if $\lambda \in \omega_t + S_{\theta_t} = \{\omega_t + z;\ z\in\mathbf{C},\ z\neq 0,\ |\arg z|<\theta_t\}$ then $\lambda\in\rho(\Lambda(t))$ and*

$$\|R(\lambda,\Lambda(t))\|_{\mathscr{L}(E)} \leqq \frac{M_t}{|\lambda-\omega_t|}.$$

(II) *For each $t\in[0,T]$, $D_{\Lambda(t)}=F$ and $\|\cdot\|_F$ is equivalent to $\|\cdot\|_{D_{\Lambda(t)}}$.*

(III) *The function $\Lambda: t\to\Lambda(t)$ belongs to $C^\alpha(0,T;\mathscr{L}(F,E))$ $(0<\alpha<1)$.*

In the following proposition we shall prove that $\|\cdot\|_{D_{\Lambda(t)}}$ is equivalent to $\|\cdot\|_F$, uniformly for $t\in[0,T]$.

PROPOSITION A.1. *Let* (I)–(III) *hold. Then there exist γ_1 and γ_2 such that for each $x\in F$ and $t\in[0,T]$ we have*

$$\gamma_1\|x\|_F \leqq \|x\| + \|\Lambda(t)x\| \leqq \gamma_2\|x\|_F. \tag{A.1}$$

PROOF. Given $t_0\in[0,T]$ from (II) we deduce the existence of α_0 and β_0 such that for each $x\in F$, $\alpha_0\|x\|_F \leqq \|x\|+\|\Lambda(t_0)x\| \leqq \beta_0\|x\|_F$. Let $\delta_0>0$ be such that if $|t-t_0|<\delta_0$ then $\|\Lambda(t)-\Lambda(t_0)\|_{\mathscr{L}(F,E)} < \alpha_0/2$; hence

$$\|x\|_{D_{\Lambda(t)}} \leqq \|x\| + \|\Lambda(t_0)x\| + \|\Lambda(t)x - \Lambda(t_0)x\| \leqq (\beta_0 + \alpha_0/2)\|x\|_F$$

and

$$\|x\|_{D_{\Lambda(t)}} \geqq \|x\| + \|\Lambda(t_0)x\| - \|\Lambda(t)x - \Lambda(t_0)x\| \geqq \frac{\alpha_0}{2}\|x\|_F.$$

By a compactness argument we obtain (A.1).

The next result shows that in condition (I) the numbers ω_t, M_t and θ_t can be chosen independent of $t \in [0, T]$.

Proposition A.2. *Let* (I)–(III) *hold. Then there exist* $\omega \in \mathbf{R}$, $M > 0$ *and* $\theta \in]\pi/2, \pi]$ *such that if* $\lambda \in \omega + S_\theta$ *then* $\lambda \in \rho(\Lambda(t))$ *and*

$$\|R(\lambda, \Lambda(t))\|_{\mathscr{L}(E)} \leqq \frac{M}{|\lambda - \omega|}. \tag{A.2}$$

Proof. Given $t_0 \in [0, T]$, let $\bar{\omega}_0 \in \mathbf{R}$, $\bar{M}_0 > 0$ and $\theta_0 \in]\pi/2, \pi]$ be such that if $\lambda \in \bar{\omega}_0 + S_{\theta_0}$ then $\lambda \in \rho(\Lambda(t_0))$ and

$$\|R(\lambda, \Lambda(t_0))\|_{\mathscr{L}(E)} \leqq \frac{\bar{M}_0}{|\lambda - \bar{\omega}_0|}.$$

Let us choose $\omega_0 > \bar{\omega}_0$: for each $\lambda \in \omega_0 + S_{\theta_0}$ we get from Proposition A.1

$$\begin{aligned}\gamma_1\|R(\lambda, \Lambda(t_0))\|_{\mathscr{L}(E,F)} &\leqq \|R(\lambda, \Lambda(t_0))\|_{\mathscr{L}(E, D_{\Lambda(t_0)})}\\ &\leqq \|R(\lambda, \Lambda(t_0))\|_{\mathscr{L}(E)} + \|\lambda R(\lambda, \Lambda(t_0)) - 1\|_{\mathscr{L}(E)}\\ &\leqq \frac{\bar{M}_0}{|\lambda - \bar{\omega}_0|} + 1 + \frac{|\lambda|\bar{M}_0}{|\lambda - \bar{\omega}_0|}.\end{aligned}$$

Hence there is $c_0 > 0$ such that for each $\lambda \in \omega_0 + S_{\theta_0}$

$$\|R(\lambda, \Lambda(t_0))\|_{\mathscr{L}(E,F)} \leqq c_0. \tag{A.3}$$

Let $\delta_0 > 0$ be such that if $|t - t_0| < \delta_0$ then $\|\Lambda(t_0) - \Lambda(t)\|_{\mathscr{L}(F,E)} < (2c_0)^{-1}$. Hence when $\lambda \in \omega_0 + S_{\theta_0}$ and $|t - t_0| < \delta_0$ there exists in $\mathscr{L}(E)$

$$R(\lambda, \Lambda(t)) = R(\lambda, \Lambda(t_0))[1 + (\Lambda(t_0) - \Lambda(t))R(\lambda, \Lambda(t_0))]^{-1}$$

and

$$\|R(\lambda, \Lambda(t))\|_{\mathscr{L}(E)} \leqq \frac{2\bar{M}_0}{|\lambda - \bar{\omega}_0|}.$$

Setting

$$M_0 = 2\bar{M}_0 \sup_{\lambda \in \omega_0 + S_{\theta_0}} \frac{|\lambda - \omega_0|}{|\lambda - \bar{\omega}_0|},$$

we proved that given $t_0 \in [0, T]$ there are $\omega_0 \in \mathbf{R}$, M_0, $\delta_0 > 0$ and $\theta_0 \in]\pi/2, \pi]$

such that for each $\lambda \in \omega_0 + S_{\theta_0}$ and $t \in]t_0 - \delta_0, t_0 + \delta_0[\cap [0, T]$ we have $\lambda \in \rho(\Lambda(t))$ and

$$\| R(\lambda, \Lambda(t)) \|_{\mathscr{L}(E)} \leqq \frac{M_0}{|\lambda - \omega_0|}.$$

Let now $t_1, \cdots, t_n \in [0, T]$ be such that $\{(t_i - \delta_i, t_i + \delta_i)\}_{i=1,\cdots,n}$ is a finite covering of $[0, T]$. By taking $\omega = \max_i \omega_i$, $\theta = \min_i \theta_i$ and

$$M = \sup \left\{ M_i \left| \frac{\lambda - \omega}{\lambda - \omega_i} \right|, 1 \leqq i \leqq n, \lambda \in \omega + S_\theta \right\}$$

the conclusion follows.

REMARK A.3. From the proof it is obvious that if $\omega_t < 0$ for each $t \in [0, T]$ then $\omega < 0$.

PROPOSITION A.4. *Let* (I)–(III) *hold and let* ω, θ *be given by Proposition* A.2. *For each* $\omega_1 > \omega$ *there exist* $K_i > 0$ $(i = 1, 2, 3)$ *such that for each* $\lambda \in \omega_1 + S_\theta$ *and* $t, s \in [0, T]$ *we have*:

$$\| (\Lambda(s) - \Lambda(t)) R(\lambda, \Lambda(s)) \|_{\mathscr{L}(E)} \leqq K_1 |t - s|^\alpha, \tag{A.4}$$

$$\| R(\lambda, \Lambda(t)) - R(\lambda, \Lambda(s)) \|_{\mathscr{L}(E)} \leqq \frac{K_2}{|\lambda - \omega|} |t - s|^\alpha, \tag{A.5}$$

$$\| R(\lambda, \Lambda(t)) - R(\lambda, \Lambda(s)) \|_{\mathscr{L}(E,F)} \leqq K_3 |t - s|^\alpha. \tag{A.6}$$

If moreover $\Lambda \in h^\alpha(0, T; \mathscr{L}(F, E))$ *then for given* $\varepsilon > 0$ *there is* $\delta_\varepsilon > 0$ *such that if* $|t - s| < \delta_\varepsilon$ *then* (A.4), (A.5) *and* (A.6) *hold with* $K_1 = K_2 = K_3 = \varepsilon$.

PROOF. In a similar fashion as in the proof of (A.3) we deduce that given $\omega_1 > \omega$ there is $c_1 > 0$ such that

$$\| R(\lambda, \Lambda(t)) \|_{\mathscr{L}(E,F)} \leqq c_1 \tag{A.7}$$

for each $\lambda \in \omega_1 + S_\theta$ and $t \in [0, T]$, hence from (III) we get (A.4). As we can write

$$R(\lambda, \Lambda(t)) - R(\lambda, \Lambda(s)) = R(\lambda, \Lambda(t))(\Lambda(t) - \Lambda(s)) R(\lambda, \Lambda(s))$$

from (A.2) and (A.4) we get (A.5) and from (A.4) and (A.7) we obtain (A.6). The last part of the proof is obvious.

By virtue of Propositions A.2 and A.4 we can state the following

PROPOSITION A.5. *By substituting* (*if necessary*) $\Lambda(t)$ *by* $\Lambda(t) - \delta I$ (*where* δ *is a real number such that* $\delta > \omega$) *we can suppose that* (IV) *of section* 1 *is verified* *and that the estimates* (A.4)–(A.6) *hold with* $\omega_1 < 0$. *In this case we have the following properties for* $\Lambda(t)$:

$$\text{(A.8)}\qquad \begin{gathered} \text{if } \operatorname{Re}\lambda \geqq 0, \text{ then } \lambda \in \rho(\Lambda(t)) \text{ and} \\ \|R(\lambda, \Lambda(t))\|_{\mathcal{L}(E)} \leqq \frac{M_1}{1+|\lambda|} \qquad \text{for each } t \in [0, T], \end{gathered}$$

$$\text{(A.9)}\qquad \|\Lambda(t)\Lambda^{-1}(s) - 1\|_{\mathcal{L}(E)} \leqq K_1 |t-s|^{\alpha} \quad \text{for } t, s \in [0, T],$$

$$\text{(A.10)}\qquad \text{if } \lambda \in S_\theta \text{ and } t, s \in [0, T], \text{ then } \lambda \in \rho(\Lambda(t)),$$

$$\|R(\lambda, \Lambda(t))\|_{\mathcal{L}(E)} \leqq \frac{M}{|\lambda|} \quad \text{and} \quad \|R(\lambda, \Lambda(t)) - R(\lambda, \Lambda(s))\|_{\mathcal{L}(E)} \leqq \frac{K_2}{|\lambda|}|t-s|^{\alpha}.$$

PROOF. To prove (A.8) it is sufficient to note that, as $\omega < 0$, we have

$$\sup_{\operatorname{Re}\lambda \geqq 0} \frac{1+|\lambda|}{|\lambda - \omega|} < \infty;$$

hence (A.8) is a consequence of (A.2). (A.9) follows from (A.4) with $\lambda = 0$. (A.10) derives from (A.2) and (A.5) since

$$\sup_{\lambda \in S_\theta} \frac{|\lambda|}{|\lambda - \omega|} < \infty.$$

In section 4 we used the following results:

PROPOSITION A.6. *For each* $t \in [0, T]$, *let* $\Lambda(t)$ *be the generator of an analytic semigroup* $\xi \to e^{\Lambda(t)\xi}$ *and let* (IV) *of section* 1 *hold; then there are constants* M_k $(k = 0, 1, \cdots)$ *and* K_4 *such that for each* $t, s \in [0, T]$ *and* $\xi \geqq 0$ *we have*

$$\text{(A.11)}_k\qquad \|\xi^k \Lambda^k(t) e^{\Lambda(t)\xi}\|_{\mathcal{L}(E)} \leqq M_k \qquad (k = 0, 1, \cdots),$$

$$\text{(A.12)}\qquad \|e^{\Lambda(t)\xi} - e^{\Lambda(s)\xi}\|_{\mathcal{L}(E)} \leqq K_4 |t-s|^{\alpha}.$$

From this it follows that $(\xi, t, x) \to e^{\Lambda(t)\xi}x$ *is continuous from* $\mathbf{R}_+ \times [0, T] \times E$ *to* E.

PROOF. Let us recall (see for instance Kato [6]) that for $t, \xi > 0$ we have

$$\text{(A.13)}\qquad e^{\Lambda(t)\xi} = \frac{1}{2\pi i}\int_\gamma e^{\lambda\xi} R(\lambda, \Lambda(t))d\lambda = \frac{1}{2\pi i}\int_\gamma e^z R\left(\frac{z}{\xi}, \Lambda(t)\right)\frac{dz}{\xi}$$

where $\gamma = \gamma_+ \cup \gamma_- \cup \gamma_\varepsilon$, $\gamma_\pm = \{\lambda; \lambda = \rho e^{\pm i\theta_1}, \rho \geqq \varepsilon\}$, $\theta_1 \in [\pi/2, \theta]$, $\varepsilon > 0$ and $\gamma_\varepsilon = \{\lambda; \lambda = \varepsilon e^{i\varphi}, |\varphi| \leqq \theta_1\}$. From (A.10) it follows that

$$\|e^{\Lambda(t)\xi}\|_{\mathcal{L}(E)} \leqq \frac{M}{\pi}\int_\varepsilon^{+\infty} e^{\rho\cos\theta_1}\frac{d\rho}{\rho} + \frac{M}{2\pi}\int_{-\theta_1}^{\theta_1} e^{\varepsilon\cos\varphi}d\varphi$$

from which we obtain $(A.11)_0$. In addition, from (A.13) we get

$$\Lambda(t)e^{\Lambda(t)\xi} = \frac{1}{2\pi i}\int_\gamma e^{\lambda\xi}\Lambda(t)R(\lambda,\Lambda(t))d\lambda$$

$$= \frac{1}{2\pi i}\int_\gamma \lambda e^{\lambda\xi}R(\lambda,\Lambda(t))d\lambda - \frac{1}{2\pi i}\int_\gamma e^{\lambda\xi}d\lambda$$

$$= \frac{1}{2\pi i}\int_\gamma \lambda e^{\lambda\xi}R(\lambda,\Lambda(t))d\lambda.$$

By using (A.10)

$$\|\Lambda(t)e^{\Lambda(t)\xi}\|_{\mathscr{L}(E)} \leqq \frac{M}{\pi}\int_\varepsilon^{+\infty} e^{\xi\rho\cos\theta_1}d\rho + \frac{M}{2\pi}\int_{-\theta_1}^{\theta_1} \varepsilon e^{\varepsilon\xi\cos\varphi}d\varphi$$

hence for $\varepsilon \to 0$

$$\|\Lambda(t)e^{\Lambda(t)\xi}\|_{\mathscr{L}(E)} \leqq \frac{M}{\pi}\int_0^{+\infty} e^{\xi\rho\cos\theta_1}d\rho = \frac{M}{\pi|\cos\theta_1|\xi}$$

which gives $(A.11)_1$. The estimates $(A.11)_k$ for $k > 1$ can be proved by induction. To prove (A.12) we can use (A.13) and obtain by virtue of (A.10)

$$\|e^{\Lambda(t)\xi} - e^{\Lambda(s)\xi}\| \leqq \left[\frac{K_2}{\pi}\int_\varepsilon^{+\infty} e^{\rho\cos\theta_1}\frac{d\rho}{\rho} + \frac{K_2}{2\pi}\int_{-\theta_1}^{\theta_1} e^{\varepsilon\cos\varphi}d\varphi\right]|t-s|^\alpha$$

from which (A.12) follows. To get the last part of the theorem, it is sufficient to use (A.12).

PROPOSITION A.7. *Let the assumptions of Proposition* A.5 *hold. Then the function* $t \to \Lambda^{-1}(t)$ *is in* $C^\alpha(0, T; \mathscr{L}(E, F))$. *If in addition* $t \to \Lambda(t)$ *is in* $h^\alpha(0, T; \mathscr{L}(F, E))$, *then* $t \to \Lambda^{-1}(t)$ *is in* $h^\alpha(0, T; \mathscr{L}(E, F))$.

PROOF. As we can suppose $\omega < 0$, it is sufficient to put $\lambda = 0$ in (A.6).

PROPOSITION A.8. *The hypotheses on* $\Lambda(t)$ *given by* (I)–(IV) *of section* 1 *are equivalent to the following assumptions* (*of Tanabe* [19] *and Sobolevski* [16]):

(A) *For each* $t \in [0, T]$, $\Lambda(t)$ *is a closed linear operator in* E *with a domain* F *dense in* E *and independent of* t. *Set* $\|\cdot\|_F = \|\cdot\|_{D_{\Lambda(0)}}$.

(B) *For each* $t \in [0, T]$, *the resolvent set of* $\Lambda(t)$ *contains the half plane* $\operatorname{Re}\lambda \geqq 0$ *and*

$$\|R(\lambda, \Lambda(t))\|_{\mathscr{L}(E)} \leqq \frac{M_1}{1+|\lambda|}, \qquad \operatorname{Re}\lambda \geqq 0, \quad t \in [0, T]. \tag{A.14}$$

(C) *There exist* $0 < \alpha < 1$ *and* $K_1 > 0$ *such that for* $\operatorname{Re}\lambda \geqq 0$ *and* $t, s \in [0, T]$:

(A.15) $$\|(\Lambda(s)-\Lambda(t))R(\lambda,\Lambda(s))\|_{\mathscr{L}(E)} \leqq K_1|t-s|^\alpha.$$

PROOF. Let us first prove that we can obtain (I) and (IV) from (A) and (B). Let (A.14) hold and choose $p_1 \in]0, \inf(1, M_1)[$. Set $m = p^{-1}\sqrt{M_1^2 - p^2}$ for each $p \in]0, p_1]$.

If $\lambda \in S = \{\lambda = x + iy; mx + |y| + 1 = 0\}$ then there is $\lambda_0 = iy_0$ verifying $|\lambda - \lambda_0| = \operatorname{dist}(\lambda_0, S)$ so that

(A.16) $$|\lambda - \lambda_0| = \frac{1+|y_0|}{\sqrt{1+m^2}} = \frac{p(1+|\lambda_0|)}{M_1},$$

hence $|\lambda - \lambda_0| \, \|R(\lambda_0, \Lambda(t))\|_{\mathscr{L}(E)} \leqq p < 1$. From this follows $\lambda \in \rho(\Lambda(t))$ and by virtue of (A.16)

$$\|R(\lambda,\Lambda(t))\|_{\mathscr{L}(E)} \leqq \frac{\|R(\lambda_0,\Lambda(t))\|}{1-p} \mathscr{L}(E) \leqq \frac{M_1}{1-p} \cdot \frac{1+|\lambda_0|+|\lambda-\lambda_0|}{1+|\lambda_0|} \cdot \frac{1}{1+|\lambda|}$$
$$\leqq \frac{M_1+p}{1-p} \cdot \frac{1}{1+|\lambda|},$$

hence

(A.17) $$\|R(\lambda,\Lambda(t))\|_{\mathscr{L}(E)} \leqq \frac{M_1'}{1+|\lambda|} \qquad \text{where } M_1' = \frac{M_1+p_1}{1-p_1}.$$

As $M_1' > M_1$, (A.17) holds also for $\operatorname{Re}\lambda \geqq 0$.

Now we can conclude that (I) and (IV) are true with $\omega = -p_1/\sqrt{M_1^2 - p_1^2}$, $\theta \in]\pi/2, \pi]$ such that $\operatorname{tg}\theta = \omega^{-1}$ and

$$M = \sup_{\lambda \in \omega + S_\theta} \frac{M_1'|\lambda - \omega|}{1+|\lambda|}.$$

Let us deduce now (II) and (III) from (A), (B) and (C). Given $s \in [0, T]$ and $y \in F$, set $x = \Lambda(s)y$. From (C) with $\lambda = 0$ and $t \in [0, T]$ we get

(A.18) $$\|\Lambda(s)y - \Lambda(t)y\| \leqq k_1|t-s|^\alpha \|\Lambda(s)y\|.$$

Now

(A.19) $$\|y\|_{D_{\Lambda(t)}} \leqq \|y\| + \|\Lambda(s)y\| + \|\Lambda(s)y - \Lambda(t)y\| \leqq (1 + K_1T^\alpha)\|y\|_{D_{\Lambda(s)}}$$

from which (II) follows. Finally, by using (A.18) and (A.19) we obtain:

$$\|\Lambda(s)y - \Lambda(t)y\| \leqq K_1|t-s|^\alpha \|y\|_{D_{\Lambda(s)}} \leqq K_1|t-s|^\alpha(1 + k_1T^\alpha)\|y\|_F,$$

i.e. condition (III).

To prove that (I)–(IV) imply (A)–(B)–(C) it is sufficient to use (A.8) and (A.4).

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Scuola Normale Superiore
Pisa, Italy

Istituto Matematico "Castelnuovo"
Università di Roma, Italy